## 1 Definitions

Orthogonal polynomials are orthogonal with respect to a certain function, known as the weight function $w(x)$, and a defined interval. The weight function must be continuous and positive such that its moments $\mu_n$ exist.

$$\mu_n := \int_a^b w(x)x^n \, dx, \ n = 0, 1, 2, \ldots$$

The interval may be infinite.

We now define the inner product of two polynomials as follows

$$\langle f, g \rangle_{w(x)} := \int_{-\infty}^{\infty} w(x)f(x)g(x) \, dx$$
We will drop the subscript indicating the weight function in future cases. Thus, as always, a sequence of polynomials \( \{p_n(x)\}_{n=0}^{\infty} \) with \( \text{deg}(p_n(x)) = n \) are called orthogonal polynomials for a weight function \( w \) if
\[
\langle p_m, p_n \rangle = h_n \delta_{mn}
\]
Above, the delta function is the Kronecker Delta Function.

There are two possible normalisations: If \( h_n = 1 \ \forall n \in \{0, 1, 2, \ldots\} \), the sequence is orthonormal. If the coefficient of highest degree term is 1 for all elements in the sequence, the sequence is monic. We note that orthogonal polynomials are thus linearly independent and generate bases for all polynomials of any arbitrary degree.

Important Immediate Properties:
1. \( \langle Q, p_n \rangle_w(x) = c_k p_n \) and \( \langle Q, p_n \rangle_w(x) \neq 0 \implies Q \) has a term of degree \( n \).

Question 1. Can you have two distinct sets of orthogonal polynomials corresponding to the same weight function but different intervals? What about the same interval? Is there any relation between the two distinct sets? Also, do all weight functions have corresponding sequence of orthogonal polynomials? If not, what are the defining properties of the weight functions that do?

What additional restrictions do you need on \( w(x) \) to make the inner product space complete?

Answers: For two sets of orthogonal polynomials corresponding to the same weight function but different intervals, look at the example set and the Legendre Polynomials.

For two sets of orthogonal polynomials corresponding to the same weight function but the same interval, look at the two kinds of Chebyshev Polynomials.

2 Example 1

We choose the weight function to be a constant (say \( c \)), and choose the interval \((0,a)\). We can construct the sequence through the Gram-Schmidt process (by moving from the basis \( \{1, x, x^2, x^3, \ldots\} \) to the orthogonal basis). We start with the sequence \( \{1, x, x^2, \ldots\} \). We choose \( p_0(x) = 1 \) for now, we can later normalise the sequence to be orthonormal. Thus by the Gram-Schmidt Process,

**Proof.**
\[
p_1(x) = x - \text{proj}_{p_0(x)}x = x - \frac{\langle x, p_0(x) \rangle}{\langle p_0(x), p_0(x) \rangle} p_0(x) = x - \frac{\langle x, 1 \rangle}{\langle 1, 1 \rangle} = x - \frac{a}{2}
\]

Since, \( \langle x, 1 \rangle = \int_0^a c \cdot x \ dx = \frac{ca^2}{2} \) and \( \langle 1, 1 \rangle = \int_0^a c dx = ca \)

\[
\implies p_2(x) = x^2 - \text{proj}_{p_0(x)}x^2 - \text{proj}_{p_1(x)}x^2 = x^2 - \frac{\langle x^2, p_0(x) \rangle}{\langle p_0(x), p_0(x) \rangle} p_0(x) - \frac{\langle x^2, p_1(x) \rangle}{\langle p_1(x), p_1(x) \rangle} p_1(x) =
\]

\[
x^2 - \frac{\langle x^2, 1 \rangle}{\langle 1, 1 \rangle} - \frac{\langle x^2, x - \frac{a}{2} \rangle}{\langle x - \frac{a}{2}, x - \frac{a}{2} \rangle} \left(x - \frac{a}{2}\right) = x^2 - \frac{ca^3}{3} \frac{1}{ca} - \frac{12}{ca^3} \left(x - \frac{a}{2}\right) = x^2 - \frac{a^2}{3} - ax + \frac{a^2}{2} = x^2 - ax + \frac{a^2}{6}
\]
as,
\[
\langle x^2, 1 \rangle = \int_0^a c \cdot x^2 \ dx = \frac{ca^3}{3}
\]
\[ \langle x^2, (x-a/2) \rangle = \int_0^a cx^2 \cdot (x-a/2) \, dx = \frac{ca^4}{4} - \frac{ca^3}{6} = \frac{ca^4}{12} \]

\[ \langle (x-a/2), (x-a/2) \rangle = \int_0^a c(x-a/2)^2 \, dx = \frac{ca^3}{3} - \frac{ca^2}{2} + \frac{ca^3}{4} = \frac{ca^3}{12} \]

And similarly by continuing the Gram-Schmidt process, we get that

\[ p_3(x) = x^3 - \frac{3a}{2} x^2 + \frac{3a^2}{5} x - \frac{a^3}{20}, \quad p_4(x) = x^4 - 2ax^3 + \frac{9a^2}{7} x^2 - \frac{2a^3}{7} x + \frac{a^4}{70} \]

\[ p_5(x) = x^5 - \frac{5a}{2} x^4 + \frac{20a^2}{9} x^3 - \frac{5a^3}{6} x^2 + \frac{5a^4}{42} x - \frac{a^5}{252} \]

The orthonormal polynomials would be \( q_n(x) = p_n(x) / \sqrt{h_n} \), where \( h_n = \int_0^a c p_n(x)^2 \, dx \)

**Question 2.** What is the general expression for \( p_n(x) \) in the above example? What about \( h_n(x) \)? Is this a complete inner product space?

3 Three-term Recurrence Relation

**Theorem 1.** A sequence of orthogonal polynomials \( \{p_n(x)\}_{n=0}^\infty \) satisfies

\[ p_{n+1}(x) = (A_n x + B_n)p_n(x) + C_n p_{n-1}(x), \quad n = 1, 2, 3... \]
where,
\[ A_n = \frac{k_{n+1}}{k_n}, \quad n = 0, 1, 2, 3... \text{ and } C_n = -\frac{A_n}{A_{n-1}} \cdot \frac{h_n}{h_{n-1}}, \quad n = 1, 2, 3... \]
where \( k_n \) is the leading coefficient of \( p_n(x) \)

**Proof.** We first note that the degree of \( p_{n+1}(x) \) is the same as the degree of \( xp_n(x) \). Thus, by defining the coefficients of the leading terms of each polynomial, we note that the leading term of \( p_{n+1}(x) \) is \( k_{n+1}x^{n+1} \), which is equal to the leading term of \( \frac{k_{n+1}}{k_n}xp_n := A_nxp_n \).

\[ \implies \langle p_{n+1}, xp_n(x) \rangle = A_n \quad (1) \]

Now we consider the lower order terms. \( p_{n+1} - A_nxp_n \) is a polynomial of degree \( \leq n \) and

\[ p_{n+1} - A_nxp_n = \sum_{k=0}^{n} c_kp_k(x) \quad (2) \]

This is because orthogonal polynomials are the basis for all polynomials. The orthogonality property now gives us that,

\[ \langle p_{n+1}(x) - A_nxp_n(x), p_k(x) \rangle = \sum_{m=0}^{n} c_m\langle p_m(x), p_k(x) \rangle = c_k\langle p_k(x), p_k(x) \rangle = h_kc_k \]

\[ = \langle p_{n+1}(x), p_k(x) \rangle - \langle A_nxp_n(x), p_k(x) \rangle = -A_n\langle p_n(x), xp_k(x) \rangle \]

\[ \implies c_k = -\frac{A_n}{h_k}\langle p_n(x), xp_k(x) \rangle \]

But, \( \langle p_n(x), xp_k(x) \rangle = 0 \) for \( k < n - 1 \). Thus, \( \forall k < n - 1, c_k = 0 \) Therefore, from (2)

\[ \implies p_{n+1} - A_nxp_n = -A_n \left( \frac{1}{h_n}\langle p_n(x), xp_n(x) \rangle \cdot p_n(x) + \frac{1}{h_{n-1}}\langle p_n(x), xp_{n-1}(x) \rangle \cdot p_{n-1}(x) \right) \]

Or, equivalently

\[ p_{n+1}(x) - A_nxp_n(x) = c_np_n(x) + c_{n-1}p_{n-1}(x), \quad n = 1, 2, 3... \]

Further, by using the same argument as (1), we have that

\[ h_{n-1}c_{n-1} = -A_n\langle p_n(x), xp_{n-1}(x) \rangle = -A_n\frac{k_{n-1}}{k_n}h_n \implies c_{n-1} = -\frac{A_n}{A_{n-1}} \cdot \frac{h_n}{h_{n-1}} \]

\[ \square \]

The term \( B_n \) in Th. 1 can be represented as \( -\frac{A_n}{h_n} \langle p_n(x), xp_n(x) \rangle \) but the inner product cannot be put in terms of coefficients as simply as the other terms. Also, we note that if we represent the polynomials as monic polynomials, we get the simpler representation,

\[ p_{n+1}(x) = xp_n(x) + B_np_n(x) + C_np_{n-1}(x) \text{ where } C_n = -\frac{h_n}{h_{n-1}}, \quad n = 1, 2, 3... \]

If they are orthogonal polynomials, we get the representation,

\[ p_{n+1}(x) = \frac{h_{n+1}}{h_n} ((A_nx + B_n)p_n(x) + C_np_{n-1}(x)) \text{ where } C_n = -\frac{A_n}{A_{n-1}}, \quad n = 1, 2, 3... \]

**Question 3.** Which is more computationally viable, the Gram-Schmidt Process or the Three-Term Recurrence Relation? Is there any other algorithm to do it better (like FFT)?
4 Christoffel-Darboux Formula

As a consequence of the Three-Term Recurrence Relation, we have that

**Theorem 2.** A sequence of orthogonal polynomials \( \{p_n(x)\}_{n=0}^{\infty} \) satisfies

\[
\sum_{k=0}^{n} \frac{p_k(x)p_k(y)}{h_k} = \frac{k_n}{h_n k_{n+1}} \cdot \frac{p_{n+1}(x)p_n(y) - p_{n+1}(y)p_n(x)}{x - y}, \quad n = 0, 1, 2...
\]

and as \( y \to x \)

\[
\sum_{k=0}^{n} \frac{p_k(x)^2}{h_k} = \frac{k_n}{h_n k_{n+1}} \cdot (p_{n+1}'(x)p_n(x) - p_{n+1}(x)p_n'(x)), \quad n = 0, 1, 2...
\]

**Proof.** The three term recurrence relation implies that

\[
p_{n+1}(x)p_n(y) = (A_n x + B_n)p_n(x)p_n(y) + C_n p_{n-1}(x)p_n(y), \quad n = 1, 2, 3...
\]

and

\[
p_{n+1}(y)p_n(x) = (A_n x + B_n)p_n(y)p_n(x) + C_n p_{n-1}(y)p_n(x), \quad n = 1, 2, 3...
\]

Subtracting the two equations above, we have

\[
p_{n+1}(x)p_n(y) - p_{n+1}(y)p_n(x) = A_n (x - y)p_n(x)p_n(y) + C_n [p_{n-1}(x)p_n(y) - p_{n-1}(y)p_n(x)]
\]

\[
\Rightarrow A_n (x - y)p_n(x)p_n(y) = C_n p_{n-1}(x)p_n(y) - C_n p_{n-1}(y)p_n(x) - p_{n+1}(x)p_n(y) + p_{n+1}(y)p_n(x)
\]

Dividing all through by \( A_n h_n \), we get

\[
\Rightarrow \frac{(x - y)p_n(x)p_n(y)}{h_n} = -p_{n-1}(y)p_n(x) + p_{n-1}(x)p_n(y) + -p_{n+1}(y)p_n(x) + p_{n+1}(x)p_n(y)
\]

\[
\Rightarrow \sum_{k=1}^{n} \frac{(x - y)p_k(x)p_k(y)}{h_k} = \sum_{k=1}^{n} \frac{-p_{k-1}(y)p_k(x) + p_{k-1}(x)p_k(y)}{A_{k-1} h_{k-1}} + \sum_{k=1}^{n} \frac{-p_{k+1}(y)p_k(x) + p_{k+1}(x)p_k(y)}{A_k h_k}
\]

By reordering,

\[
\Rightarrow (x - y) \sum_{k=1}^{n} \frac{p_k(x)p_k(y)}{h_k} = \sum_{k=1}^{n} \frac{p_{k+1}(x)p_k(y) - p_k(x)p_{k+1}(y)}{A_k h_k} - \sum_{k=1}^{n} \frac{p_k(x)p_{k-1}(y) - p_{k-1}(x)p_k(y)}{A_{k-1} h_{k-1}}
\]

By eliminating all the common terms in the two sums on the RHS,

\[
\Rightarrow (x - y) \sum_{k=1}^{n} \frac{p_k(x)p_k(y)}{h_k} = \frac{p_{n+1}(x)p_n(y) - p_n(x)p_{n+1}(y)}{A_n h_n} - \frac{p_1(x)p_0(y) - p_0(x)p_1(y)}{A_0 h_0}
\]

\[
= \frac{p_{n+1}(x)p_n(y) - p_n(x)p_{n+1}(y)}{A_n h_n} - \frac{k_0(p_1(x)k_0 - k_0 p_1(y))}{k_1 h_0}
\]

as \( A_0 = \frac{k_1}{k_0} \) and \( p_0(x) = p_0(y) = k_0 \)

Further, \( p_1(x) - p_1(y) = k_1 (x - y) \) \( \Rightarrow \frac{k_0(p_1(x)k_0 - k_0 p_1(y))}{k_1 h_0} = \frac{k_1^2(x - y)}{h_0} = (x - y) \frac{p_0(x)p_0(y)}{h_0} \)

\[
\Rightarrow (x - y) \sum_{k=1}^{n} \frac{p_k(x)p_k(y)}{h_k} = \frac{p_{n+1}(x)p_n(y) - p_n(x)p_{n+1}(y)}{A_n h_n}
\]
\[ \sum_{k=0}^{n} \frac{p_k(x)p_k(y)}{h_k} = \frac{k_n}{(x-y)h_{n+1}}(p_{n+1}(x)p_n(y) - p_n(x)p_{n+1}(y)) \]

Thus, we prove the Christoffel-Darboux formula. To prove the confluent form, we take the limit as \( y \to x \)

\[ \lim_{y \to x} \frac{p_{n+1}(x)p_n(y) - p_n(x)p_{n+1}(y)}{x-y} = \lim_{x \to y} \frac{p_n(x)(p_{n+1}(x) - p_{n+1}(y)) - p_{n+1}(x)(p_n(x) - p_n(y))}{x-y} \]

\[ = p_n(x)p_{n+1}' - p_{n+1}(x)p_n'(x) \]

\[ \square \]

5 Zeros

**Theorem 3.** If \( \{p_n(x)\}_{x=0}^{\infty} \) is a sequence of orthogonal polynomials on the interval \((a,b)\) with respect to the weight function \( w(x) \), then the polynomial \( p_n(x) \) has exactly \( n \) real simple zeros on the interval \((a,b)\)

**Proof.** Since the degree of \( p_n(x) \) is \( n \), the number of distinct real roots is at most \( n \). Suppose that \( p_n(x) \) has \( m \leq n \) distinct real roots \( x_1, x_2, ..., x_m \) in \((a,b)\) of odd multiplicity (we ignore the roots of even multiplicity). Then, the polynomial

\[ p_n(x)(x-x_1)(x-x_2)...(x-x_m) \]

does not change sign on the interval \((a,b)\) as each root occurs an even number of times in the interval \((a,b)\). Thus, as \( w(x) > 0 \forall x \in (a,b) \)

\[ \int_{a}^{b} w(x)p_n(x)(x-x_1)(x-x_2)...(x-x_m)dx \neq 0 \]

But, by orthogonality this integral equals 0 if \( m < n \). Thus, \( m = n \), which implies that \( p_n(x) \) has \( n \) distinct real zeros in \((a,b)\). \( \square \)

This can be seen in the roots of the previous example. We can also see in that example that the roots of the polynomials are interlacing.

**Theorem 4.** If \( \{p_n(x)\}_{x=0}^{\infty} \) is a sequence of orthogonal polynomials on the interval \((a,b)\) with respect to the weight function \( w(x) \), then the zeros of \( p_n(x) \) and \( p_{n+1}(x) \) separate each other.

**Proof.** This follows from the confluent form of the Christoffel-Darboux formula, stated as:

\[ \sum_{k=0}^{n} \frac{p_k(x)^2}{h_k} = \frac{k_n}{h_nk_{n+1}} \cdot (p_{n+1}'(x)p_n(y) - p_n(x)p_{n+1}'(y)), \quad n = 0, 1, 2... \]

We note that

\[ h_n = \int_{a}^{b} w(x)\{p_n(x)\}^2 dx > 0, \quad n = 0, 1, 2... \]

as \( w(x) > 0 \forall x \).

\[ \implies \frac{k_n}{h_nk_{n+1}} \cdot (p_{n+1}'(x)p_n(x) - p_n(x)p_{n+1}'(x)) = \sum_{k=0}^{n} \frac{p_k(x)^2}{h_k} > 0 \]
Now, suppose $x_{n,k}$ and $x_{n,k+1}$ are two consecutive zeros of $p_n(x)$ with $x_{n,k} < x_{n,k+1}$. Since all $n$ zeros of $p_n(x)$ are real and distinct, $p'_n(x_{n,k})$ and $p'_n(x_{n,k+1})$ should have opposite signs (as $p_n(x_{n,k}) = 0$ and $p_n(x_{n,k+1}) = 0$, but there are no zeros in between $x_{n,k}$ and $x_{n,k+1}$ by definition, hence the curve must be increasing in one direction and decreasing in the other). Hence we have that

$$p_n(x_{n,k}) = 0 = p_n(x_{n,k+1}) \text{ and } p'_n(x_{n,k}) \cdot p'_n(x_{n,k+1}) < 0$$

Then, as per the formula above,

$$k_n \cdot (p_{n+1}'(x_{n,k})) > 0 \text{ and } k_n \cdot (p_{n+1}'(x_{n,k+1})) > 0$$

$$\Rightarrow \left( \frac{k_n}{k_{n+1}} \right)^2 \cdot (p_{n+1}'(x_{n,k})) \cdot (p_{n+1}'(x_{n,k+1})) > 0$$

$$\Rightarrow (p_{n+1}(x_{n,k})) \cdot (p_{n+1}(x_{n,k+1})) > 0$$

$$\Rightarrow p_{n+1}(x_{n,k}) \cdot p_{n+1}(x_{n,k+1}) < 0 \text{ as } p'_n(x_{n,k}) \cdot p'_n(x_{n,k+1}) < 0$$

Further, the continuity of $p_{n+1}(x)$ implies that there should be an odd number of zeros of $p_{n+1}(x)$ between $x_{n,k}$ and $x_{n,k+1}$ as the sign must change between them. However, this holds for any two consecutive zeros of $p_n$. Thus, the only way this is possible is if there is one zero of $p_{n+1}$ between any two consecutive zeros of $p_n$. Therefore, $a < x_{n,k} < x_{n+1,k+1} < x_{n,k+1} < b \forall n > 0, k < n$ where $x_{n,k}$ is the $k$th root of $p_n$ and so on.

This can be seen in the following graph of roots plotted for the first five non-trivial Orthonormal Polynomials of the constant function as in the above example:

**Question 4.** Is there any relationship between the Cauchy Interlacing Theorem and Orthogonal Polynomials that lends to this property?
6 Gauss Quadrature

6.1 Lagrange Interpolation

If \( f \) is a continuous function on \((a,b)\) and \(x_1 < x_2 < x_3 < \ldots < x_n\) are \(n\) distinct points in \((a,b)\), then there exists exactly one polynomial \(P\) with degree \(\leq n-1\) such that \(P(x_i) = f(x_i)\) for all \(i = 1, 2, 3, \ldots, n\) (there are infinite polynomials of degree \(n\) or greater). This is because if there existed another polynomial, say \(Q\) with such a property, we have that \((P - Q)(x)\) has \(n\) distinct zeros, for each \(x_i\). But this is not possible as \((P - Q)(x)\) has only \(n\) zeros. The existence of the unique polynomial \(P\) can be easily demonstrated (and the polynomial found) by using Lagrange interpolation as follows. Define

\[
p(x) = (x - x_1)(x - x_2)\ldots(x - x_n)
\]

\[
\Rightarrow p'(x) = \sum_{i=1}^{n} \frac{p(x)}{(x - x_i)}
\]

\[
\Rightarrow p'(x_i) = \lim_{x \to x_i} \frac{p(x)}{x - x_i} = (x_i - x_1)(x_i - x_2)\ldots(x_i - x_{i-1})(x_i - x_{i+1})\ldots(x_i - x_n) := q_i(x_i)
\]

where \(q_i(x) := \frac{p(x)}{x - x_i}\). And consider

\[
P(x) = \sum_{i=1}^{n} f(x_i) \frac{p(x)}{(x - x_i)p'(x_i)} = \sum_{i=1}^{n} f(x_i) \left( \frac{x - x_1}{x - x_1} \right) \left( \frac{x - x_{i-1}}{x - x_1} \right) \left( \frac{x - x_{i+1}}{x - x_{i-1}} \right) \ldots \left( \frac{x - x_n}{x - x_{i+1}} \right) = \sum_{i=1}^{n} f(x_i) \frac{q_i(x)}{q_i(x_i)}
\]

We see that thus, \(P(x_i) = f(x_i)\) for all \(i = 1, 2, 3, \ldots, n\).

6.2 Gauss quadrature formula

Let \(\{p_n(x)\}_{n=0}^{\infty}\) be a sequence of orthogonal polynomials on the interval \((a,b)\) with respect to the weight function \(w(x)\). Then we take the \(n\) distinct roots of \(p_n(x) = \{x_i\}_{i=1}^{n}\) such that \(x_1 < x_2 < \ldots < x_n\). We now consider a \(f(x)\) of degree \(\leq 2n - 1\), and \(P(x)\) as the polynomial of degree \(\leq n\) such that \(P(x_i) = f(x_i)\) for all \(i = 1, 2, 3, \ldots, n\).

If \(f\) is a polynomial of degree \(\leq 2n - 1\), then \(f(x) - P(x)\) is a polynomial of degree \(\leq 2n - 1\) with at least the zeros \(x_1 < x_2 < \ldots < x_n\). Now we define

\[
f(x) = P(x) + r(x)p_n(x)
\]

where \(r(x)\) is a polynomial of degree \(\leq n - 1\). This can also be written as

\[
f(x) = \sum_{i=1}^{n} f(x_i) \frac{p(x)}{(x - x_i)p'(x_i)} + r(x)p_n(x)
\]

\[
\Rightarrow \int_{a}^{b} w(x)f(x)\,dx = \sum_{i=1}^{n} f(x_i) \int_{a}^{b} \frac{w(x)p_n(x)}{(x - x_i)p'(x_i)}\,dx + \int_{a}^{b} w(x)r(x)p_n(x)\,dx
\]

Since \(r(x)\) is of a degree \(\leq n\), the orthogonality property implies that the second integral equals 0.

\[
\Rightarrow \int_{a}^{b} w(x)f(x)\,dx = \sum_{i=1}^{n} f(x_i)\lambda_{n,i} \text{ such that } \lambda_{n,i} := \int_{a}^{b} \frac{w(x)p_n(x)}{(x - x_i)p'_n(x)}\,dx
\]
This is known as the Gauss quadrature formula, and it gives us the value of the integral in the case that \( f \) is a polynomial of degree \(<2n\) and if the values of \( f(x_i) \) is known for the \( n \) zeros \( x_1 < x_2 < x_3 \ldots x_n \) of the polynomial \( p_n(x) \). If \( f \) is not a polynomial of degree \(<2n\), this leads to an approximation of the integral:

\[
\int_a^b w(x) f(x) \, dx \approx \sum_{i=1}^{n} f(x_i) \lambda_{n,i} \text{ such that } \lambda_{n,i} := \int_a^b \frac{w(x)p_n(x)}{(x-x_i)p'_n(x)} \, dx
\]

The coefficients \( \{\lambda_{n,i}\}_{i=1}^{n} \) are called Christoffel numbers. We note that these numbers do not depend on the function \( f \).

We will now try to approximate \( \sin(x) \) using this formula and the set of orthogonal polynomials used above. We found the roots of the polynomial, and found the respective Christoffel numbers.

This graph shows the five Christoffel numbers for the 5th orthonormal polynomial for the weight function described above.

Further, I used the fifth orthonormal polynomial of the constant weight function to approximate some functions, and found that for a random polynomial of degree 9, and for other functions such as \( \sin(x) \) and \( \exp(x) \), the Gauss Quadrature was able to approximate the function with a near 100% precision (the error is thought to be due to the errors in numerical integration).

This is incredible, as after the initial steps of computing the zeros of the orthonormal polynomial, and computing its Christoffel numbers, the integration of any function will only take \( 2n \) steps (one for computing the function at each point, and then to add it perform summation), where \( n \) is the order of the orthonormal polynomial. And we have seen that we achieve near 100% accuracy even with just a degree 5 polynomial.

```matlab
1 % Christoffel Approximation
2 pol=[1; -5/2;+20/9; -5/6;+5/42; -1/252]; % The orthonormal polynomial is defined by its coefficients
3 xi=roots(pol); % MATLAB finds the roots of the polynomial. (This is the first source of numerical error)
4 p5=@(x) x.^5-5/2*x.^4+20/9*x.^3-5/6*x.^2+5/42*x-1/252; % The orthonormal polynomial defined as a function in MATLAB
5 dp5=@(x) 5*x.^4-4*5/2*x.^3+3*20/9*x.^2-2*5/6*x+5/42; % The derivative of the orthonormal polynomial defined as a function in MATLAB
```
These Christoffel numbers are all positive, as can be seen below:

\[ \lambda_{n,i} = \int_a^b w(x) l_{n,i}(x) \, dx \text{ with } l_{n,i}(x) := \frac{p_n(x)}{(x-x_i)p'_n(x)} \]

We consider \( l_{n,i}^2(x) - l_{n,i}(x) = \frac{p_n^2(x)}{(x-x_i)^2p'_n(x)^2} - \frac{p_n(x)}{(x-x_i)p'_n(x)} \) which is a polynomial of degree \( < 2n - 2 \) which vanishes at the zeros \( \{x_{n,k}\}_{k=1}^n \) of \( p_n(x) \). Hence,

\[ l_{n,i}^2(x) - l_{n,i}(x) = p_n(x)s(x) \]

for some polynomial \( q(x) \) of degree \( \leq n - 2 \). Note: \( q(x) \) is actually equal to

\[ \frac{p_n(x)}{(x-x_i)^2p'_n(x)^2} - \frac{3}{(x-x_i)p'_n(x)} \]

\[ \implies \int_a^b w(x)(l_{n,i}^2(x) - l_{n,i}(x)) \, dx = \int_a^b w(x)p_n(x)q(x) \, dx = 0 \]

by orthogonality. Hence we have,

\[ \lambda_{n,i} = \int_a^b w(x)l_{n,i}(x) \, dx = \int_a^b w(x)\{l_{n,i}(x)\}^2 \, dx > 0 \]

Now, we can also prove that

**Theorem 5.** If \( \{p_n(x)\}_{x=0}^\infty \) is a sequence of orthogonal polynomials on the interval \( (a,b) \) with respect to the weight function \( w(x) \), then we have that between any two zeros of \( p_m(x) \) there is at least one zero of \( p_n(x) \), for any \( m < n \).

**Proof.** Suppose that \( x_{m,k} \) and \( x_{m,k+1} \) are two consecutive zeros of \( p_m(x) \) and that there is no zero of \( p_n(x) \) in \( (x_{m,k}, x_{m,k+1}) \). Then consider the polynomial

\[ g(x) = \frac{p_m(x)}{(x-x_{m,k})(x-x_{m,k+1})} \]

Then we have

\[ g(x)p_m(x) \geq 0 \text{ for } x \notin (x_{m,k}, x_{m,k+1}) \]

as then \( (x-x_{m,k}) \) and \( (x-x_{m,k+1}) \) have the same sign and their product is positive. Now the Gauss quadrature formula gives

\[ \int_a^b w(x)g(x)p_m(x) \, dx = \sum_{i=1}^n \lambda_{n,i}g(x_{n,i})p_m(x_{n,i}) \]
where \(\{x_{n,i}\}_{i=1}^n\) are the zeros of \(p_n(x)\). Since we assumed that there are no zeros of \(p_n(x)\) in \((x_{m,k}, x_{m,k+1})\) we conclude that \(g(x_{n,i}p_m(x_{n,i})) > 0\) for all \(i=1,2...n\). Further, we have \(\lambda_{n,i} > 0\) for all \(i=1,2...n\). Which implies that the sum at the RHS cannot vanish. However, the integral at the left-hand side is zero by orthogonality (as we are only considering \(m < n\)). This contradiction proves that there should be at least one zero of \(p_n(x)\) between any two consecutive zeros of \(p_m(x)\). This can again be seen in the graph of the roots of the Orthogonal Polynomials.

7 Classical Orthogonal Polynomials

7.1 Hermite Polynomials

The weight function for this set of orthonormal polynomials is \(e^{-x^2}\).

The interval on which the inner product for this set of orthonormal polynomials is defined is \((-\infty, \infty)\).

The first five polynomials of this set are:

\[
\begin{align*}
2x \\
4x^2 - 2 \\
8x^3 - 12x \\
16x^4 - 48x^2 + 12 \\
32x^5 - 160x^3 + 120x
\end{align*}
\]

We plotted the first five polynomials on the interval and obtained the following graph:

![Hermite Polynomials Graph](image)

We then plotted the zeros of the first five polynomials of this set.
We also plotted the Christoffel numbers for the first five polynomials of this set.

7.2 Laguerre Polynomials

The weight function for this set of orthonormal polynomials is $e^{-x}x^\alpha$.

The interval on which the inner product for this set of orthonormal polynomials is defined is $(0, \infty)$. The first five polynomials of this set are:

$$
\begin{align*}
\alpha - x + 1 \\
\frac{3\alpha}{2} - x(\alpha + 2) + \frac{\alpha^2}{2} + \frac{x^2}{2} + 1 \\
\frac{11\alpha}{6} - x\left(\frac{\alpha^2}{2} + \frac{5\alpha}{2} + 3\right) + x^2\left(\frac{\alpha}{2} + \frac{3}{2}\right) + \alpha^2 + \frac{\alpha^3}{6} - \frac{x^3}{6} + 1 \\
\frac{25\alpha}{12} - x\left(\frac{\alpha^3}{6} + \frac{3\alpha^2}{2} + \frac{13\alpha}{3} + 4\right) - x^3\left(\frac{\alpha}{6} + \frac{2}{3}\right) + x^2\left(\frac{\alpha^2}{4} + \frac{7\alpha}{4} + 3\right) + \frac{35\alpha^2}{24} + \frac{5\alpha^3}{12} + \frac{\alpha^4}{24} + x^4 + 1
\end{align*}
$$
\[
\frac{137 \alpha}{60} + x^2 \left( \frac{\alpha^3}{12} + \alpha^2 + \frac{47 \alpha}{12} + 5 \right) + x^4 \left( \frac{\alpha}{24} + \frac{5}{24} \right) - x \left( \frac{\alpha^4}{24} + \frac{7 \alpha^3}{12} + \frac{71 \alpha^2}{24} + \frac{77 \alpha}{12} + 5 \right) - ... \\
\]

\[
x^3 \left( \frac{\alpha^2}{12} + \frac{3 \alpha}{4} + \frac{5}{3} \right) + \frac{15 \alpha^2}{8} + \frac{17 \alpha^3}{24} + \frac{\alpha^4}{8} + \frac{\alpha^5}{120} - \frac{x^5}{120} + 1
\]

We plotted the first five polynomials on the interval and obtained the following graph (for a random parameter, \( \alpha = 0.6324 \))

![Laguerre polynomials graph](image)

We then plotted the zeros of the first five polynomials of this set.

![Degree of Polynomial vs Zeros](image)

We also plotted the Christoffel numbers for the first five polynomials of this set.

![Degree of Polynomial vs Christoffel numbers](image)
7.3 Legendre Polynomials

The weight function for this set of orthonormal polynomials is 1.
The interval on which the inner product for this set of orthonormal polynomials is defined is (-1,1).
The first five polynomials of this set are:

\[
\begin{align*}
    x \\
    \frac{3x^2}{2} - \frac{1}{2} \\
    \frac{5x^3}{2} - \frac{3x}{2} \\
    \frac{35x^4}{8} - \frac{15x^2}{4} + \frac{3}{8} \\
    \frac{63x^5}{8} - \frac{35x^3}{4} + \frac{15x}{8}
\end{align*}
\]
We plotted the first five polynomials on the interval and obtained the following graph:

We then plotted the zeros of the first five polynomials of this set.

We also plotted the Christoffel numbers for the first five polynomials of this set.
7.4 Jacobi Polynomials

The weight function for this set of orthonormal polynomials is \((1 - x)\alpha(1 + x)^\beta\).

The interval on which the inner product for this set of orthonormal polynomials is defined is \((-1, 1)\).

The first two polynomials of this set are:

\[
x^2 \left( \frac{\alpha^2}{8} + \frac{\alpha \beta}{4} + \frac{7 \alpha}{8} + \frac{\beta^2}{8} + \frac{7 \beta}{8} + \frac{3}{2} \right) - \frac{\beta}{8} - \frac{\alpha \beta}{4} - \frac{\alpha^2}{8} + \frac{\beta^2}{8} + x \left( \frac{\alpha^2}{4} + \frac{3 \alpha}{4} - \frac{\beta^2}{4} - \frac{3 \beta}{4} \right) - \frac{1}{2}
\]

We plotted the first five polynomials on the interval and obtained the following graph:

We then plotted the zeros of the first five polynomials of this set (interestingly, some of them had imaginary parts).
We also plotted the Christoffel numbers for the first five polynomials of this set.

7.5 Chebyshev Polynomials of the First Kind

The weight function for this set of orthonormal polynomials is \( \frac{1}{\sqrt{1-x^2}} \). The interval on which the inner product for this set of orthonormal polynomials is defined is (-1,1). The first five polynomials of this set are:

\[
\begin{align*}
&x \\
&2x^2 - 1 \\
&4x^3 - 3x \\
&8x^4 - 8x^2 + 1 \\
&16x^5 - 20x^3 + 5x
\end{align*}
\]
We plotted the first five polynomials on the interval and obtained the following graph:

We then plotted the zeros of the first five polynomials of this set.
7.6 Chebyshev Polynomials of the Second Kind

The weight function for this set of orthonormal polynomials is $\frac{1}{\sqrt{1-x^2}}$. The interval on which the inner product for this set of orthonormal polynomials is defined is (-1,1). The first five polynomials of this set are:

\[
\begin{align*}
2x \\
4x^2 - 1 \\
8x^3 - 4x \\
16x^4 - 12x^2 + 1 \\
32x^5 - 32x^3 + 6x
\end{align*}
\]

We plotted the first five polynomials on the interval and obtained the following graph:
We then plotted the zeros of the first five polynomials of this set.

7.7 Gegenbauer polynomials

The weight function for this set of orthonormal polynomials is \((1 - x^2)^{\alpha - 1/2}\).

The interval on which the inner product for this set of orthonormal polynomials is defined is (-1,1).

The first five polynomials of this set are:

\[
\begin{align*}
2\alpha x & \\
x^2 (2\alpha^2 + 2\alpha) & - \alpha \\
x^3 \left(\frac{4\alpha^3}{3} + 4\alpha^2 + \frac{8\alpha}{3}\right) & - x \left(2\alpha^2 + 2\alpha\right) \\
\frac{\alpha}{2} - x^2 (2\alpha^3 + 6\alpha^2 + 4\alpha) + x^4 \left(\frac{2\alpha^4}{3} + 4\alpha^3 + \frac{22\alpha^2}{3} + 4\alpha\right) & + \frac{\alpha^2}{2}
\end{align*}
\]
\[(4 \alpha^5 + \frac{8 \alpha^4}{3} + \frac{28 \alpha^3}{3} + \frac{40 \alpha^2}{3} + \frac{32 \alpha}{5}) x^5 + \left(-\frac{4 \alpha^4}{3} - 8 \alpha^3 - \frac{44 \alpha^2}{3} - 8 \alpha\right) x^3 + (\alpha^3 + 3 \alpha^2 + 2 \alpha) x\]

We plotted the first five polynomials (for a random \(\alpha\) on the interval and obtained the following graph:

We then plotted the zeros of the first five polynomials of this set.

We also plotted the Christoffel numbers for the first five polynomials of this set.